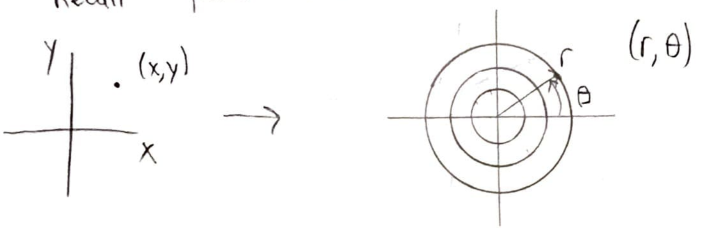
Lecture 19

Polar Coordinates

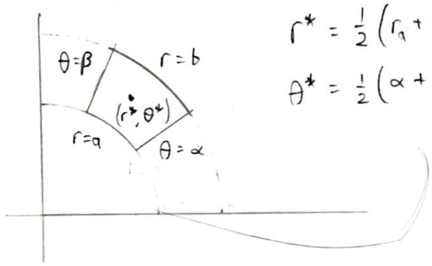
Recall polar coordinates.



To convert from cartesian to polar coordinates:

$$r^2 = x^2 + y^2, \qquad x = r\cos(\theta), \qquad y = r\sin(\theta).$$

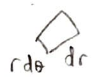
Polar Rectangle



$$r^* = \tfrac{1}{2}(r_a + r_b)$$
$$\theta^* = \tfrac{1}{2}(\alpha + \beta)$$

In, the infinitesmal limit,

$$\text{Area} \approx r\,dr\,d\theta$$



Double Integral in Polar Coordinates

If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\},$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta.$$

If f is continuous on a polar region of the form

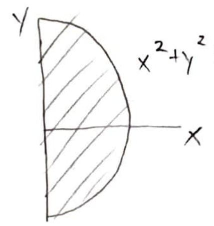
$$D = \{(r,\theta) \mid \emptyset \le a < r \le b,\ \alpha \le \theta \le \beta\}$$

then,

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$$

Ex. 1

Find $\iint_R x \, dA$ where R is the region described by $x^2+y^2 \le 9$ with $x \ge 0$.



$r = 0$ to $r = 3$

$\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$

$$\int_{-\pi/2}^{\pi/2} \int_0^3 r\cos(\theta) r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^3 r^2 \cos(\theta) \, dr \, d\theta$$

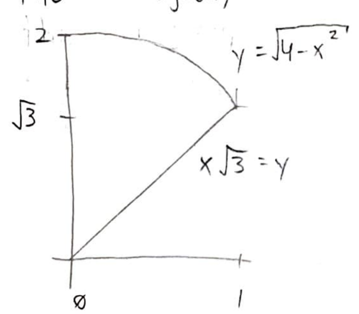
$$\int_{-\pi/2}^{\pi/2} \frac{1}{3} r^3 \Big|_0^3 \cos(\theta) \, d\theta = 9 \int_{-\pi/2}^{\pi/2} \cos(\theta) \, d\theta$$

$$9 \left[\sin(\theta) \Big|_{-\pi/2}^{\pi/2} \right] = 9 [1 - -1] = 18$$

Ex. 2

Evaluate $\displaystyle\int_0^1 \int_{x\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2} \, dy \, dx$

Draw the region,



$r = 0$ to $r = 2$

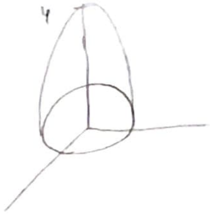
$\theta = \frac{\pi}{3}$ to $\theta = \frac{\pi}{2}$

$$\int_0^2 \int_{\pi/3}^{\pi/2} r \, r \, d\theta \, dr = \int_0^2 \int_{\pi/3}^{\pi/2} r^2 \, d\theta \, dr$$

$$\int_{\pi/3}^{\pi/2} \frac{1}{3} r^3 \Big|_0^2 \, d\theta = \frac{8}{3} \int_{\pi/3}^{\pi/2} d\theta = \frac{8}{3} \left[\frac{\pi}{2} - \frac{\pi}{3} \right]$$

Ex. 3

Let D be the region bounded above by $z = 4 - x^2 - y^2$ and below by the xy-plane. Find the volume of D.



$r = 0$ to $r = 2$

$\theta = 0$ to $\theta = 2\pi$

$$\iint z \, dA = \text{Volume}$$

$$\int_0^{2\pi} \int_0^{2} (4 - x^2 - y^2) \, dA = \int_0^{2\pi} \int_0^{2} (4 - r^2) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{2} (4r - r^3) \, dr$$

$$= 2\pi \left[2r^2 - \frac{1}{4} r^4 \Big|_0^2 \right] = 2\pi \left[8 - 4 \right]$$

$$= 8\pi$$